Progress from May 2021-April 2022

Our main focus so far was to learn foundational material related to the theory of representations of p-adic groups, the subject area of our interest. As a part of this, we undertook several reading projects.

- **Representatons of p-adic groups:** We studied complex representations of ℓ-groups, which are locally compact Hausdorff groups that have a fundamental system of neighborhoods of the identity consisting of compact and open subgroups. After looking into the generalities in the subject like construction of Haar measure on an ℓ-group and its quotient by a closed subgroup, the notion of smooth and admissible representations and their basic properties, we narrowed down to the study of representations of $\mathrm{GL}_n(F)$ where F is a non-Archimedean local field.

  We began by getting a preliminary understanding of the structure of $\mathrm{GL}_n(F)$, like its parabolic subgroups, their levi components and unipotent radicals, Bruhat, Cartan, Iwasawa and Iwahori decompositions. Analogous to the case of finite groups where a representation is seen to be a module over the group algebra, the corresponding concept of Hecke algebra was looked into. We then studied the properties of two fundamental functors namely smooth induction(mostly from a parabolic subgroup) and Jacquet functor. The smooth irreducible representations of $\mathrm{GL}_n(F)$ are classified into two categories: one that arise via induction from a proper parabolic subgroup and the other which does not(called cuspidal representations). Harish Chandra gave equivalent characterisations of cuspidal representations in terms of its Jacquet module and matrix coefficients. Using these facts, it was proved that every smooth irreducible representation of general linear group over a local field is admissible.

  We then looked into some results for $\mathrm{GL}_2(F)$. A theorem of Bernstein and Zelevinski asserts that if p is a continuous map between two locally compact totally disconnected spaces X and Y, $\mathcal{F}$ a sheaf on X and G a group acting on X and $\mathcal{F}$, existence of non-zero distributions on X imples the existence of non-zero distributions on atleast one of the fibres of p. By aptly choosing the space Y, one can restrict to the study of distributions on smaller spaces which proved to be extremely useful. This was used to establish many results which were true for $\mathrm{GL}_2(\mathbb{F}_q)$ in the case of $\mathrm{GL}_2(F)$. The first was to realise the contragradient of an irreducible admissible representation (π, V) of $\mathrm{GL}_2(F)$ in the same space V. Secondly, to prove that the dimension of the space of Whittaker functionals of an irreducible admissible representation of $\mathrm{GL}_2(F)$ is atmost one. Consequently, a twisted Jaquet module of an irreducible admissible representation of $\mathrm{GL}_2(F)$ is atmost of dimension one.

  Currently, we are trying to understand the classification of irreducible admissible representations of $\mathrm{GL}_n(F)$ using the theory of segments. The Zelevinski classification says that any irreducible admissible representation of $\mathrm{GL}_n(F)$ can be seen as the unique subrepresentation of a parabolically induced representation from a multisegment satisifying certain conditions. The Langland's classification gives a similar result in terms of a unique quotient.

- **Tate's thesis:**

  As an introduction to the theory of automorphic forms, we read Tate's thesis. Prior to Tate, Hecke had attached an L-series to an idele class character. Tate reproved the meromorphic continuation and functional equation of Hecke L-function using fourier analysis. As a consquence, analytic continuation and functional equations of some well known functions like Riemann zeta function, Dedekind zeta function etc were obtained. This approach also threw light on the appearance of gamma functions in the fuctional equations of these classical zeta functions.

We began with the study of Pontryagin dual of locally compact groups and their properties. This was applied to understand in particular the dual of local fields and the adele group of global fields. The dual of a local field is seen to be topologically isomorphic to itself via a non-trivial additive character which is fixed. A measure is chosen which is self dual in the sense of fourier inversion and the absolute value is normalised so that it becomes the module of automorphism of this self dual measure. The multiplicative Haar measure is also suitably defined and a proof of factorisation of quasi-characters of the multiplicative group of non-zero elements was obtained. It is shown that the fourier transform gives an isomorphism on the Schwartz-Bruhat finctions on local fields. These observations are used to define the local zeta integral and prove its meromorphic continuation. A certain functional equation was also obtained in terms of local L-factors which coincided with the terms in the Euler product factorisation of Hecke L-function.

Global Schwartz Bruhat functions on adeles is defined to be the space generated by those functions that are factorizable into a product of local Schwartz Bruhat functions. Similar to the case of local fields, adeles of a global field are seen to be topologically isomorphic to its dual and a self dual Haar measure was constructed. Global zeta function is defined and is observed to be the product of corresponding local zeta functions. Its meromorphic continuation and a functional equation is obtained by using Riemann-Roch theorem.